CHAPTER 3

FINITE DIFFERENCE SCHEMES

Finite difference methods are a very general purpose numerical scheme whose theory and development have been described clearly in almost every numerical analysis book. For these details, readers are advised to consult any book on numerical methods to their liking. We used Refs [1–3] which helped us a great deal in developing the finite difference method, Ref. [4] for solving underwater acoustic wave propagation problems. Although the general theory of finite difference methods with regard to consistency, stability and convergence is not detailed in this monograph, we describe fully the theory of the finite difference scheme developed to solve propagation problems. We begin with a set of symbol definitions, and some basic concepts which provide the basis for the development of the finite difference scheme.

Let us partition the rectangular propagation domain we consider into a set of small rectangular blocks (Fig. 3.1). We use the index m to indicate the vertical direction, and the index n to indicate the horizontal direction. The wave field, indicated by u, is a function of (r, z). At the point (n, m) it means $u(r, z) = u(n\Delta r, m\Delta z) = u_m^n$. A lower case k is used to indicate Δr, and lower case h to indicate Δz. The forward finite difference of the wave field is the difference between two neighboring points, e.g. $u^{n+1} - u^n$; otherwise in the reverse direction, it is the backward difference, e.g. $u^{n-1} - u^n$. By the central difference, we mean $u^{n+1} - u^{n-1}$ as the forward central difference, $u^{n-1} - u^{n+1}$ as the backward central difference; both differences regard u^n as the center point. We use δ_z to represent the central difference operator in the z-direction, D is the finite difference operator in general.

Mathematically speaking, δ_z operating on the function $F(z)$ means

$$\delta_z f = f(z + \tfrac{1}{2}h) - f(z - \tfrac{1}{2}h). \tag{3.1}$$

To establish a relationship between the operators δ_z and D, we expand both $f(z + \frac{1}{2}h)$ and $f(z - \frac{1}{2}h)$ in powers of h and substitute them into equation (3.1), obtaining

$$\begin{aligned}
\delta_z f &= f(z + \tfrac{1}{2}h) - f(z - \tfrac{1}{2}h) \\
&= \left[f(z) + \tfrac{1}{2}hf'(z) + \frac{1}{2!}\frac{1}{4}h^2 f''(z) + \frac{1}{3!}\frac{1}{8}h^3 f'''(z) + \cdots\right] \\
&\quad - \left[f(z) - \frac{1}{2}hf'(z) + \frac{1}{2!}\frac{1}{4}h^2 f''(z) - \frac{1}{3!}\frac{1}{8}h^3 f'''(z) + \cdots\right], \\
&= 2\left(\frac{h}{2}f'(z) + \frac{1}{3!}\left(\frac{h}{2}\right)^3 f'''(z) + \cdots\right), \\
&= 2\sinh\left(\frac{h}{2}\frac{\partial}{\partial z}\right)f.
\end{aligned} \tag{3.2}$$

The general finite difference operator D we just introduced is simply $\partial/\partial z$ in the z-direction. From formula (3.2), we have

$$\partial_z f = 2\sinh\left(\frac{h}{2}D\right)f, \tag{3.3}$$

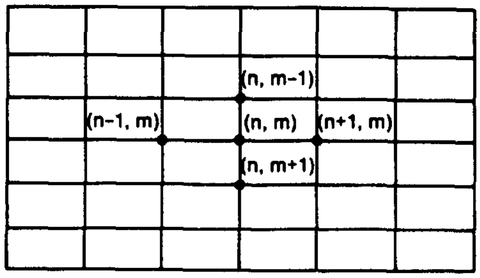

Fig. 3.1. Rectangular propagation domain.

which gives

$$D = \frac{2}{h}\sinh^{-1}\left(\frac{\delta_z}{2}\right). \tag{3.4}$$

Therefore,

$$D^2 = \frac{\delta_z^2}{h^2}\left(1 - \tfrac{1}{12}\delta_z^2 + \tfrac{1}{90}\delta_z^4 - \cdots\right). \tag{3.5}$$

3.1. FORMULATION

We use the following general expression to express the standard parabolic wave equation:

$$\begin{aligned}\frac{\partial}{\partial r}u &= a(k_0, r, z)u + b(k_0, r, z)u_{zz},\\ &= Lu,\end{aligned} \tag{3.6}$$

where

$$L = a(k_0, r, z) + b(k_0, r, z)\frac{\partial^2}{\partial z^2},$$

$$a(k_0, r, z) = \frac{i}{2}k_0(n^2(r, z) - 1),$$

$$b(k_0, r, z) = \frac{i}{2k_0}.$$

Using a Taylor expansion for $u(r + k, z)$ one obtains

$$\begin{aligned}u(r + k, z) &= \left(1 + k\frac{\partial}{\partial r} + \frac{1}{2!}k^2\frac{\partial^2}{\partial r^2} + \cdots\right)u(r, z),\\ &= \exp\left(k\frac{\partial}{\partial r}\right)u(r, z).\end{aligned} \tag{3.7}$$

If we write $z = mh$, $r = nk$ and $u(r, z) = u(nk, mh) = u_m^n$, and use expression (3.7) to solve expression (3.6) and retain only the second order difference, an *explicit* formula is obtained:

$$u_m^{n+1} = \left(1 + k\frac{\partial}{\partial r}\right)u_m^u = \left(1 + a(k_0, r, z)k + \frac{b}{h^2}k\delta_z^2\right)u_m^n. \tag{3.8}$$

Using the second order central difference for δ_z^2, in equation (3.8), one finds that

$$u_m^{n+1} = (1 + a_m^n k)u_m^n + \frac{b_m^2}{h^2}k(u_{m+1}^n - 2u_m^n + u_{m-1}^n). \tag{3.9}$$

This is an explicit finite difference equation by which the field values at a range $r = (n + 1)\,k$ can be generated from a knowledge only of the field at the previous range step.

Alternatively we may split expression (3.7) in such a way that

$$\exp\left(-\frac{1}{2}k\frac{\partial}{\partial r}\right)u_m^{n+1} = \exp\left(\frac{1}{2}k\frac{\partial}{\partial r}\right)u_m^n. \tag{3.10}$$

Expanding the exponential series on both sides of expression (3.10) and retaining only the linear terms, we obtain

$$\left(1-\frac{1}{2}k\frac{\partial}{\partial r}\right)u_m^{n+1} = \left(1+\frac{1}{2}k\frac{\partial}{\partial r}\right)u_m^n. \tag{3.11}$$

Since $\partial/\partial r = a(k_0, r, z) + b(k_0, r, z)\partial^2/\partial z^2$, equation (3.11) becomes

$$\left[1-\frac{1}{2}k\left(a(k_0,r,z)+b(k_0,r,z)\frac{\partial^2}{\partial z^2}\right)\right]u_m^{n+1} = \left[1+\frac{1}{2}k\left(a(k_0,r,z)+b(k_0,r,z)\frac{\partial^2}{\partial z^2}\right)\right]u_m^n. \tag{3.12}$$

Using the first term of equation (3.5) for D^2, and substituting it into equation (3.12), gives

$$\left[1-\frac{1}{2}k\left(a(k_0,r,z)+b(k_0,r,z)\frac{1}{h^2}\delta_z^2\right)\right]u_m^{n+1} = \left[1+\frac{1}{2}k\left(a(k_0,r,z)+b(k_0,r,z)\frac{1}{h^2}\delta_z^2\right)\right]u_m^n. \tag{3.13}$$

Substituting for δ_z^2 in equation (3.13) and writing $s = k/h^2$, we obtain

$$\begin{aligned}&[1-\tfrac{1}{2}k(a(k_0,r,z)+b(k_0,r,z)s)]u_m^{n+1} - \tfrac{1}{2}b(k_0,r,z)s(u_{m+1}^{n+1}+u_{m-1}^{n+1})\\ &\qquad = [1+\tfrac{1}{2}k(a(k_0,r,z)+b(k_0,r,z)s)]u_m^n + \tfrac{1}{2}b(k_0,r,z)s(u_{m+1}^n+u_{m-1}^n).\end{aligned} \tag{3.14}$$

Equation (3.14) is an *implicit* finite difference equation known conventionally as the Crank–Nicolson scheme.

Now, we have developed an explicit scheme (3.9), and an implicit scheme (3.14). The explicit scheme does not need information on the next advanced level but usually requires a small step size for stability. The implicit scheme is, on the other hand, unconditionally stable in most applications. Despite the complicated dependence of equation (3.14) on the field at the next $(n+1)$ range level, this equation may also be solved given the field only at the previous range and the necessary boundary conditions at $z=0$ and $z=z_B$.

These forms are particularly amenable to numerical computation because of the limited storage space required. From the foregoing discussion it is apparent that to solve the parabolic equation (PE), we need to know the field at some initial range, and that we must also impose the appropriate boundary conditions at the surface and bottom of our rectangular propagation domain. Boundary information is needed at the present range level for the explicit scheme, and needed both at the present and the next advanced range levels for the implicit scheme. We exhibit solutions for both explicit scheme (3.9) and implicit schemes (3.14) to show how the boundary points are handled. For a wave propagating in a rectangular region, the boundary points are described by $u(r_0, z_0)$, $u(r_1, z_0)$, $u(r_0, z_B)$ and $u(r_1, z_B)$. The first two points are the surface boundary points, the last two points are the bottom boundary points. Using the first term of expression (3.5) to relate D^2 to δ_z^2 and substituting it into expression (3.9), we find expression (3.9) in a matrix form:

$$\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{m-1}^{n+1} \\ u_m^{n+1} \end{bmatrix} = \begin{bmatrix} \alpha_1-2\beta_1 & \beta_1 & 0 & \cdots & 0 & 0 \\ \beta_2 & \alpha_2-2\beta_2 & \beta_2 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & \alpha_{m-1}-2\beta_{m-1} & \beta_{m-1} \\ 0 & 0 & 0 & \cdots & \beta_m & \alpha_m-2\beta_m \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{m-1}^n \\ u_m^n \end{bmatrix} + \begin{bmatrix} \beta_1 u_0^n \\ 0 \\ \vdots \\ 0 \\ \beta_m u_{m+1}^n \end{bmatrix}, \tag{3.15}$$

where

$$1 + a(k_0, r_n, z_m) = \alpha_m,\ b_m^n k/h^2 = \beta_m.$$

It is seen that the first component of the vector on the r.h.s. of expression (3.15) u_0^n is the surface boundary point. The last component of the same vector u_{m+1}^n is the bottom boundary point.

Let us now turn our attenuation to the implicit scheme. We already related D^2 to δ_z^2 and now substitute it into expression (3.13) using the same definition, written below equation (3.15), for β_m^n

and $\alpha_m^n = 1 - \frac{1}{2}ka_m^n$. In addition, define

$$\begin{aligned} \gamma_m^n &= 1 + \tfrac{1}{2}ka_m^n, \\ X_m^{n+1} &= 1 - \tfrac{1}{2}ka_m^{n+1} + b_m^{n+1}s, \\ Y_m^n &= \gamma_m^n - \beta_m^n = 1 + \tfrac{1}{2}ka_m^n - b_m^n s. \end{aligned} \tag{3.16}$$

Using the above definitions, equation (3.14) can be expressed in a matrix form

$$\begin{bmatrix} X_1 & -\frac{1}{2}\beta_1^{n+1} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2}\beta_2^{n+1} & X_2 & -\frac{1}{2}\beta_2^{n+1} & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & X_{m-1} & -\frac{1}{2}\beta_{m-1}^{n+1} \\ 0 & 0 & 0 & \cdots & -\frac{1}{2}\beta_m^{n+1} & X_m \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{m-1}^{n+1} \\ u_m^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\beta_1^{n+1}u_0^{n+1} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\beta_m^{n+1}u_{m+1}^{n+1} \end{bmatrix}$$

$$+ \begin{bmatrix} Y_1 & \frac{1}{2}\beta_1^n & 0 & \cdots & 0 & 0 \\ \frac{1}{2}\beta_2^n & Y_2 & \frac{1}{2}\beta_2^n & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & Y_{m-1} & \frac{1}{2}\beta_{m-1}^n \\ 0 & 0 & 0 & \cdots & \frac{1}{2}\beta_m^n & Y_m \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{m-1}^n \\ u_m^n \end{bmatrix} - \begin{bmatrix} \frac{1}{2}\beta_1^n u_0^n \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\beta_m^n u_{m+1}^n \end{bmatrix}. \tag{3.17}$$

If the surface and bottom boundary values are known at the advanced level, equation (3.17) may be solved for the field at the advanced level by inverting the matrix operator on the l.h.s. of this equation. In the next chapter, we will address in detail the specification of boundary conditions. In the remainder of this chapter we will proceed to develop the theory of consistency, stability and convergence for both the explicit scheme (3.9) and the implicit scheme (3.14).

3.2. CONSISTENCY

The conventional definition of consistency states that a finite difference approximation to the parabolic wave equation is consistent if

$$\frac{\text{truncation error}}{k} \to 0 \text{ as } h, k \to 0.$$

This definition relates the finite difference operator to the true operator. Therefore, consistency would mean the finite difference operator is consistent with the true operator. This relationship is the definition of consistency in the sense of Keller [5].

Recall expression (3.6) exhibits the parabolic wave equation in an operator form. We define the true operator as follows.

Let

$L_h^1[u; h, k]$ be the finite difference operator for the explicit scheme (3.9) such that

$$L_h^1[u; h, k] = u_m^{n+1} - (1 + a_m^n k)u_m^n + \frac{b_m^n}{h^2}k(u_{m+1}^n - 2u_m^n + u_{m-1}^n). \tag{3.18}$$

and

$L_h^2[u; h, k]$ be the finite difference operator for the implicit scheme (3.14) such that

$$\begin{aligned} L_h^2[u; h, k] = {} & [1 - \tfrac{1}{2}ka(k_0, (n+1)k, mh) + b(k_0, (n+1)k, mh)s]u_m^{n+1} \\ & - \tfrac{1}{2}b(k_0, (n+1)k, mh)s(u_{m+1}^{n+1} + u_{m-1}^{n+1}) \\ & - [1 + \tfrac{1}{2}ka(k_0, nk, mh) + b(k_0, nk, mh)s]u_m^n \\ & + \tfrac{1}{2}b(k_0, nk, mh)s(u_{m+1}^n + u_{m-1}^n). \end{aligned} \tag{3.19}$$

Define

$$\tau_j[u; h, k] = L[u] - L_h^j[u'h, k], \quad j = 1,2. \tag{3.20}$$

If

$$\lim_{k\to 0} \tau_j[u; h, k] = 0 \text{ for } j = 1, 2,$$

we say that the method is consistent, meaning that the finite difference operator is consistent with the true operator for both explicit and implicit schemes. Prior to the development of the concept of consistency, it is necessary to examine the "initial local discretization error," usually called the "truncation error."

Expanding u_m^{n+1}, u_{m+1}^n and u_{m-1}^n upon u_m^n, we find

$$u_m^{n+1} = u_m^n + k\left(\frac{\partial u}{\partial r}\right)_m^n + \frac{k^2}{2!}\left(\frac{\partial^2 u}{\partial r^2}\right)_m^n + \cdots, \tag{3.21}$$

$$u_{m+1}^n = u_m^n + h\left(\frac{\partial u}{\partial z}\right)_m^n + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial z^2}\right)_m^n + \cdots, \tag{3.22}$$

and

$$u_{m-1}^n = u_m^n - h\left(\frac{\partial u}{\partial z}\right)_m^n + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial z^2}\right)_m^n - \cdots. \tag{3.23}$$

Substituting expressions (3.21), (3.22) and (3.23) into expression (3.9), we obtain

$$\begin{aligned} &u_m^{n+1} - (1 + a_m^n k)u_m^n - \frac{b_m^n}{h^2}k(u_{m+1}^n - 2u_m^n + u_{m-1}^n) \\ &= \left\{k\left(\frac{\partial u}{\partial r}\right)_m^n - k a_m^n u_m^n - k b_m^n\left(\frac{\partial^2 u}{\partial z^2}\right)_m^n\right\} \\ &+ \left[\frac{k^2}{2}\left(\frac{\partial^2 u}{\partial r^2}\right)_m^n - \frac{1}{12} b_m^n k h^2\left(\frac{\partial^4 u}{\partial z^4}\right)_m^n + \frac{k^3}{3!}\left(\frac{\partial^3 u}{\partial r^3}\right)_m^n + \cdots\right]. \end{aligned} \tag{3.24}$$

The terms inside the { } of expression (3.24) = 0 because they satisfy expression (3.6). Let $E[e]$ indicate the principal part of the initial discretization error of expression (3.24). Then, we have

$$E[e] = 0(k^2 + kh^2).$$

Then,

$$\tau_1[u; h, k] = L[u] - L_h^1[u; h, k] = -E[e] - \text{high order terms}. \tag{3.25}$$

It is easily seen from formula (3.24) that

$$\lim_{h,k\to 0} \tau_1[u; h, k] = \lim_{n,k\to 0} \{-E[e] - \text{high order terms}\} = 0,$$

therefore, the explicit scheme (3.9) is consistent. Next, we examine the consistency for the implicit scheme (3.14).

In addition to formulas (3.21)–(3.23), we need to expand u_{m+1}^{n+1}, and u_{m-1}^{n+1} upon u_m^n:

$$\begin{aligned} u_{m+1}^{n+1} &= u_m^{n+1} + h\left(\frac{\partial u}{\partial z}\right)_m^{n+1} + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial z^2}\right)_m^{n+1} + \cdots \\ &= u_m^n + k\left(\frac{\partial u}{\partial r}\right)_m^n + \frac{k^2}{2!}\left(\frac{\partial^2 u}{\partial r^2}\right)_m^n + \cdots \\ &+ h\left(\frac{\partial u}{\partial z}\right)_m^n + hk\left(\frac{\partial^2 u}{\partial z \partial r}\right)_m^n + h\frac{k^2}{2!}\left(\frac{\partial^3 u}{\partial z \partial r^2}\right) + \cdots \\ &+ \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial z^2}\right)_m^n + \frac{h^2}{2!}k\left(\frac{\partial^3 u}{\partial z^2 \partial r}\right)_m^n + \frac{h^2}{2!}\frac{k^2}{2!}\left(\frac{\partial^4 u}{\partial z^2 \partial r^2}\right)_m^n + \cdots \\ &+ \cdots; \end{aligned} \tag{3.26}$$

$$
\begin{aligned}
u_{m-1}^{n+1} &= u_m^{n+1} - h\left(\frac{\partial u}{\partial z}\right)_m^{n+1} + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial z^2}\right)_m^{n+1} - \cdots \\
&= u_m^n + k\left(\frac{\partial u}{\partial r}\right)_m^n + \frac{k^2}{2!}\left(\frac{\partial^2 u}{\partial r^2}\right)_m^n + \cdots \\
&\quad - h\left(\frac{\partial u}{\partial z}\right)_m^n - hk\left(\frac{\partial^2 u}{\partial z\partial r}\right)_m^n - h\frac{k^2}{2!}\left(\frac{\partial^3 u}{\partial z\partial r^2}\right)_m^n - \cdots \\
&\quad + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial z^2}\right)_m^n + \frac{h^2}{2!}k\left(\frac{\partial^3 u}{\partial z^2\partial r}\right)_m^n + \frac{h^2}{2!}\frac{k^2}{2!}\left(\frac{\partial^4 u}{\partial z^2\partial r^2}\right)_m^n + \cdots \\
&\quad + \cdots.
\end{aligned} \tag{3.27}
$$

Substituting expressions (3.21), (3.22), (3.23), (3.26) and (3.27) into expression (3.14), we obtain

$$
\begin{aligned}
&\{-\tfrac{1}{2}\beta_m^{n+1}u_{m+1}^{n+1} + (\alpha_m^{n+1} + \beta_m^{n+1})u_m^{n+1} - \tfrac{1}{2}\beta_m^{n+1}u_{m-1}^{n+1}\} \\
&- \{\tfrac{1}{2}\beta_m^n u_{m+1}^n + (\gamma_m^n - \beta_m^n u_m^n) + \tfrac{1}{2}\beta_m^n u_{m-1}^n\} \\
&= \left\{-kau_m^n + k\left(\frac{\partial u}{\partial r}\right)_m^n - bsh^2\left(\frac{\partial^2 u}{\partial z^2}\right)_m^n\right\} \\
&+ \left[-\frac{1}{2}kak\left(\frac{\partial u}{\partial r}\right)_m^n + \frac{k^2}{2!}\left(\frac{\partial^2 u}{\partial r^2}\right)_m^n - bs\frac{h^2}{2!}k\left(\frac{\partial^3 u}{\partial z^2\partial r}\right)_m^n\right] \\
&+ \left(-\frac{1}{2}kak\left(\frac{\partial^2 u}{\partial r^2}\right)_m^n + \frac{k^3}{4}\left(\frac{\partial^3 u}{\partial r^3}\right)_m^n - bs\frac{h^2}{2!}\frac{k^2}{2!}\left(\frac{\partial^4 u}{\partial z^2\partial r^2}\right)_m^n\right) \\
&- \frac{k^3}{12}\left(\frac{\partial^3 u}{\partial r^3}\right)_m^n - bs\frac{h^4}{12}\left(\frac{\partial^4 u}{\partial z^4}\right)_m^n + \cdots.
\end{aligned} \tag{3.28}
$$

We drop the superscripts and subscripts on a and b because b is both range and depth independent; a is assumed to be slowly varying in range. The terms inside the { } of the r.h.s. of expression (3.28) = 0 because they satisfy equation (3.6). We see that the terms inside the [ ] and the ( ) of the r.h.s. of equation (3.28) vanish because of the properties of $a(k_0, r, z)$ and $b(k_0, r, z)$ plus the terms inside the [ ] satisfy $u_{rr} - au_r - bu_{zzr} = 0$ and the terms inside the ( ) satisfy $u_{rrr} - au_{rr} - bu_{zzrr} = 0$.

Now, we want to examine whether or not

$$\lim_{h,k\to 0} \tau_2[u; h, k] \to 0.$$

Let $E[I]$ indicate the principal part of the initial descretization error of expression (3.28).

Then,

$$\tau_2[u; h, k] = L[u] - L_h^2[u; h, k] = -E[I] - \text{high order terms}. \tag{3.29}$$

It can be verified easily that

$$\lim_{h,k\to 0} \tau_2[u; h, k] = \lim_{h,k\to 0} \{-E[I] - \text{high order terms}\} = -0,$$

therefore, the implicit scheme (3.14) is consistent.

3.3. STABILITY

The general concept of stability states that the difference between the theoretical and numerical solutions remains bounded as the range step n increases, provided the range increment k remains fixed for all space steps m. To find out whether a method is stable or not, we examine the satisfaction of the stability condition; this condition can be derived by means of familiar methods such as Von Neumann's, the matrix, or the Fourier series.

We shall first derive the stability condition for the explicit scheme (3.9). We apply Von Neumann's criterion of stability to equation (3.9) by seeking a solution in the form $e^{\alpha r}e^{i\omega z}$. Using $\xi = e^{\alpha k}$ and substituting this solution into scheme (3.9), we find

$$\xi^{n+1}e^{i\omega mh} = \xi^n(1 + a_m^n k - 2\,b_m^n s)e^{i\omega h} + \xi^n b_m^n s(e^{i\omega(m+1)h} + e^{i\omega(m-1)h}). \tag{3.30}$$

Simplification of equation (3.30) gives

$$\xi = 1 + a_m^n k - 4\,sb_m^n \sin^2\left(\frac{\omega h}{2}\right).$$

$|\xi| \leqslant 1$ is required for stability, i.e.

$$\left|1 + a_m^n k - 4\,sb_m^n \sin^2\left(\frac{\omega h}{2}\right)\right| \leqslant 1. \tag{3.31}$$

When we use formula (3.9) to solve the simple heat equation, $u_r = u_{zz}$, we have

$$-1 \leqslant 1 - 4\frac{k}{h^2}\sin^2\left(\frac{\omega h}{2}\right) \leqslant 1, \tag{3.32}$$

which implies

$$\frac{k}{h^2} \leqslant \frac{1}{2}, \tag{3.33}$$

for stability.

In our case, however, $a(k_0, r, z)$ and $b(k_0, r, z)$ are functions of r and z, and are complex. We must, thus, thoroughly examine condition (3.32). With reference to condition (3.32), the r.h.s. inequality is trivially satisfied if s and b_m^n are both >0. The l.h.s. gives

$$sb_m^n \leqslant \frac{1}{2\sin^2\left(\frac{\omega h}{2}\right)}. \tag{3.34}$$

In the case where b_m^n is purely imaginary, we need

$$\left|1 - 4\,sb_m^n \sin^2\left(\frac{\omega h}{2}\right)\right| \leqslant 1. \tag{3.35}$$

Condition (3.35) does not hold for $s > 0$ and b_m^n purely imaginary; therefore, the explicit scheme (3.9) is *not stable* for problems with zero coefficient $a(k_0, r, z)$ and a purely imaginary $b(k_0, r, z)$. When $a(k_0, r, z) \neq 0$, we need inequality (3.31) to hold. In our applications, $a(k_0, r, z)$ and $b(k_0, r, z)$ are both imaginary.

Write

$$a(k_0 r, z) = ia_R,$$

$$b(k_0 r, z) = ib_R,$$

where a_R and b_R are both real. With reference to condition (3.31), we, then, have

$$\left|1 + i\left[a_R k - 4\,sb_R \sin^2\left(\frac{\omega h}{2}\right)\right]\right| \leqslant 1, \tag{3.36}$$

where the terms inside the [] are all real. Obviously, inequality (3.36) does not hold. For the inequality of expression (3.36) to hold, we must have

$$a_R k - 4\,sb_R \sin^2\left(\frac{\omega h}{2}\right) = 0,$$

which implies that the condition of stability is

$$h^2 = 4\frac{b_R}{a_R}\sin^2\left(\frac{\omega h}{2}\right). \tag{3.37}$$

Equation (3.37) holds for $h = 0$, this suggests the instability of the explicit scheme (3.9). This is the main reason why we introduced the implicit finite difference scheme (3.14).

In examining the stability of the implicit finite difference scheme, we again apply Von Neumann's method to formula (3.14). As in our treatment of the explicit scheme, we define $\xi = e^{\alpha k}$. Substituting it into expression (3.14), yields

$$-\frac{b}{2}se^{\alpha(n+1)k}e^{i\omega(m+1)h} + Xe^{\alpha(n+1)k}e^{i\omega mh} - \frac{b}{2}se^{\alpha(n+1)k}e^{i\omega(m-1)h}$$
$$= \frac{b}{2}se^{\alpha nk}e^{i\omega(m+1)h} + Ye^{\alpha nk}e^{i\omega mh} + \frac{b}{2}se^{\omega nk}e^{i\omega(m-1)h}.$$

Simplifying, we find that

$$\xi = \frac{Y + \frac{b}{2}s[2\cos(\omega h)]}{X - \frac{b}{2}s[2\cos(\omega h)]}, \tag{3.38}$$

which gives the stability condition

$$\left|\frac{1 - bs[1-\cos(\omega h)] - \frac{1}{2}ak}{1 + bs[1-\cos(\omega h)] + \frac{1}{2}ak}\right| \leqslant 1 \tag{3.39}$$

where $a(k_0, r, z)$, $b(k_0, r, z) > 0$, inequality (3.39) is satisfied for all $s > 0$; therefore, the scheme is unconditionally stable. In our applications, $a(k_0, r, z)$ and $b(k_0, r, z)$ are both purely imaginary. Specifically, we have

$$\left|\frac{1 - \frac{i}{2k_0}s[1-\cos(\omega h)] - \frac{k}{2}\left\{\frac{ik_0}{2}[n^2(r,z)-1]\right\}}{1 + \frac{i}{2k_0}s[1-\cos(\omega h)] + \frac{k}{2}\left\{\frac{ik_0}{2}[n^2(r,z)-1]\right\}}\right| \leqslant 1. \tag{3.40}$$

Equation (3.40) can be expressed in the short form

$$\left|\frac{1-iX}{1+iX}\right| \leqslant 1, \tag{3.41}$$

where

$$X = \frac{s}{2k_0}[1-\cos(\omega h)] + \frac{kk_0}{4}[n^2(r,z)-1].$$

From equation (3.41) we see that

$$\left|\frac{1-iX}{1+iX}\right| \text{ always} = 1, \quad \forall \text{ real } X.$$

Therefore, the implicit finite difference scheme, equation (3.14) is stable when it is applied to solve our parabolic wave equation.

3.4. CONVERGENCE

The convergence of the implicit finite difference scheme, equation (3.14), for solving equation (3.6) consists of finding the condition under which the difference between the theoretical solutions of the differential and finite difference equations at a fixed point (r, z), tends to zero uniformly, as the grid is partitioned in such a way that $h, k \to 0$ and indexes $m, n \to \infty$, with $mh(=z)$, $nk(=r)$

remaining fixed. To show that scheme (3.14) is convergent, we define

t.s. = the theoretical solution of equation (3.6),

n.s. = the numerical solution of equation (3.6)

and

f.s. = the finite difference solution of equation (3.6).

The norm inequality shows that

$$\|\text{t.s.} - \text{n.s.}\| \leqslant \|\text{t.s.} - \text{f.s.}\| + \|\text{f.s.} - \text{n.s.}\|.$$

Applying the consistency to the first norm of the r.h.s. and applying the stability and the error control to the second norm of the r.h.s. together establish the convergence.

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